Mechanics: Kinematics (Solutions)

FIZIKA SPhO Training

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1 Notes

1.1 Kinematic Equations

1.1.1 Introduction

Kinematics is the study of how objects move, without any analysis of how these movements come about. In kinematics, we are interested in the following quantities: displacement ($\mathbf{r}$), speed ($\mathbf{v}$), acceleration ($\mathbf{a}$) and time (t). These quantities must be measured with respect to an observer.

1.1.2 Velocity & Speed

In secondary school, you would have studied the basic 1-D kinematic equations:

$$v = u + a\,t \tag{1}$$

$$v^2 = u^2 + 2\,a\,s \tag{2}$$

$$s = u\,t + \tfrac{1}{2}a\,t^2 \tag{3}$$

To deal with 2-D or 3-D motion, the essential idea is just to break down the 2-D or 3-D motion into each axis and apply the 1-D equations. To be more proper, you can use the vector formalism introduced below.

Remark. For SPhO, your typical H2 methods (SUVAT bash along each axis) still work. Don't be intimidated by vectors - they are just a way to express multiple SUVAT equations (for each axis) into one equation!

By definition, velocity is:

$$\mathbf{v} := \frac{d\mathbf{r}}{dt} \tag{4}$$

In the integral form:

$$\mathbf{r}_f = \int_{t_0}^{t} \mathbf{v}\,dt = \mathbf{r}_0 + \int_0^t \mathbf{v}\,dt = \mathbf{r}_0 + \mathbf{v}t \tag{5}$$

Example 1.1. A particle travels with $\mathbf{v} = t\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ (this represents a constant velocity in the y-direction, and a constant acceleration in the x-direction), starting from t=0 to t=4. Find the total displacement and distance traveled.

Using equation (2), we can compute the displacement as:

$$\mathbf{r} = \int_0^4 \left(t\hat{\mathbf{i}} + 4\hat{\mathbf{j}}\right) dt = \left.\frac{t^2}{2}\right|_0^4 \hat{\mathbf{i}} + 4\,t|_0^4\,\hat{\mathbf{j}} = 8\hat{\mathbf{i}} + 16\hat{\mathbf{j}}$$

To find the net distance, we have

$$r = \sqrt{8^2 + 16^2} = 8\sqrt{5}$$

A common **but wrong** way of thinking is that since $\frac{d\mathbf{r}}{dt}$ gives the velocity $\mathbf{v}$, $\frac{dr}{dt}$ should give the speed. Instead, the speed should be computed as $|\frac{d\mathbf{r}}{dt}|$. This will be more clear when we deal with circular motion.

1.1.3 Acceleration

The definition of acceleration is:

$$\mathbf{a} := \frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) \tag{6}$$

Now, suppose that $\mathbf{a}$ is uniform, meaning the vector is constant across time: $\mathbf{a} = \mathbf{a}_0$. Then, separating variables from Equation (6),

$$d\left(\frac{d\mathbf{r}}{dt}\right) = \mathbf{a}_0\, dt \tag{7}$$

$$\int_{\mathbf{v_0}}^{\frac{d\mathbf{r}}{dt}} d\left(\frac{d\mathbf{r}}{dt}\right) = \int_0^t \mathbf{a}_0\, dt \tag{8}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_0 + \mathbf{a}_0 t \tag{9}$$

not forgetting that $\mathbf{v} = \frac{d\mathbf{r}}{dt}$. Separating variables from Equation (9) and integrating again,

$$\mathbf{r}_f = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}_0 t^2 \tag{10}$$

We can obtain another equation using this **very common trick** which will be proven later.

$$\mathbf{a} := \frac{d^2\mathbf{r}}{dt^2} = \dot{\mathbf{r}}\frac{d\dot{\mathbf{r}}}{d\mathbf{r}} \tag{11}$$

Assuming constant acceleration, $\mathbf{a} = \mathbf{a}_0$, then separating variables from Equation (11) and integrating,

$$\mathbf{a}_0 \cdot d\mathbf{r} = \dot{\mathbf{r}} \cdot d\dot{\mathbf{r}} \tag{12}$$

$$\int_{\mathbf{r}_0}^{\mathbf{r}_f} \mathbf{a}_0 \cdot d\mathbf{r} = \int_{\dot{\mathbf{r}_0}}^{\dot{\mathbf{r}_f}} \dot{\mathbf{r}} \cdot d\dot{\mathbf{r}} \tag{13}$$

$$|\mathbf{v}_f|^2 = |\mathbf{v}_0|^2 + 2\mathbf{a}_0 \cdot (\mathbf{r}_f - \mathbf{r}_0) \tag{14}$$

not forgetting that $\mathbf{v}_0 = \dot{\mathbf{r}_0}$ and $\mathbf{v}_f = \dot{\mathbf{r}_f}$.

Be *a little* careful here. Here, we are working with dot products when we "multiplied vectors" in Equation (12). When you deal with variable separation with vectors, you need to take dot products! If you are not convinced, take the 1D case of Equation (11) and work out the math in terms of scalars. You'll get the same thing as Equation (14) but in 1D.

Equations (9), (10) and (14) are simply 3-D generalizations of the SUVAT equations you are already familiar with. Once again, don't be intimidated - it is equivalent to just write the SUVAT equations along each axis.

Example 1.2. The superhero Green Lantern steps from the top of a tall building. He falls freely from rest to the ground, falling half the total distance to the ground during the last Δt seconds of his fall. What is the height h of the building?

This is a 1-D problem which involves a very basic application of the kinematic equations:

$$\frac{h}{2} = v\Delta t + \frac{1}{2}g(\Delta t)^2$$

$$v^2 = 2g\frac{h}{2} = gh$$

We can obtain the solution by solving these two simultaneous equations. You should obtain that

$$h = \left(3 + 2\sqrt{2}\right) g \left(\Delta t\right)^2$$

The algebra and manipulations in this example were certainly more tedious than setting up the equations themselves. Unfortunately, SPhO problems sometimes reduce into doing tedious amounts of algebra. Try to be comfortable with this so that you won't be stumped by anything that SPhO throws at you.

1.2 Variable Acceleration

Such questions present themselves as mathematical challenges as opposed to physical problems.

Example 1.3. When a projectile moves slowly through air, the drag is linear in the velocity, $F = -m\alpha v$. Find the velocity $v(t)$ of a projectile thrown upward at time $t = 0$ with speed v_0.

Newton's 2nd law gives:

$$m\frac{dv}{dt} = -mg - m\alpha v,$$

Be careful of the signs! The gravitational force is $-mg$, not mg, as it *decreases* the upward velocity. Likewise, the drag is $-m\alpha v$, not $+m\alpha v$, as it *decreases* the upward velocity. Draw a free-body diagram to help you visualise this!

Dividing by m and multiplying through by dt, and then integrating via separation of variables gives:

$$\int_{v_0}^{v(t)} \frac{dv}{g + \alpha v} = -\int_0^t dt.$$

$$\frac{1}{\alpha}\left[\ln(g + \alpha v)\right]_{v_0}^{v(t)} = -t.$$

Solve for v:

$$v(t) = v_0 e^{-\alpha t} + \frac{g}{\alpha}\left(e^{-\alpha t} - 1\right).$$

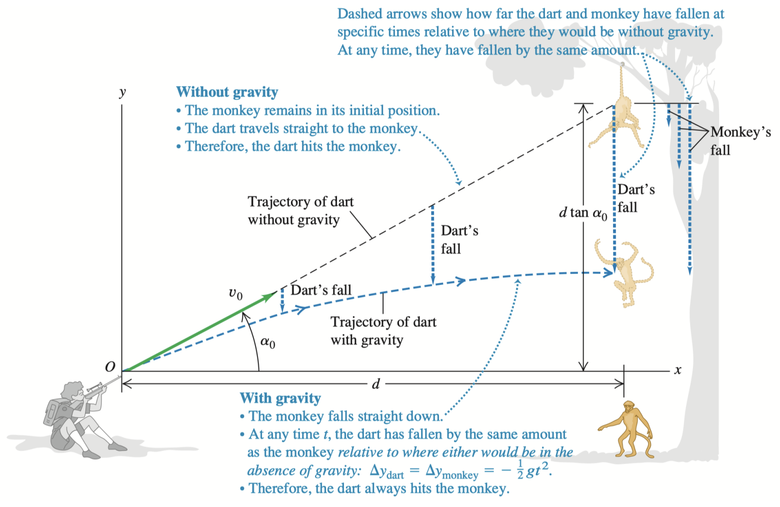
To deal with problems with variable acceleration, you need integration. This example also shows you why your **signs are important** - if for instance, you had written $+m\alpha v$ instead, v will explode to infinity as time increases, and common sense tells you this doesn't make sense!

1.3 Projectile Motion

We can analyse the motion in each direction using our SUVAT equations. For projectile motion without air drag, the **horizontal velocity remains constant** throughout the motion, while the **vertical motion is subject to a constant downward acceleration due to gravity.**

Example 1.4. A monkey escapes from the zoo and climbs a tree. After failing to entice the monkey down, the zookeeper fires a tranquilizer dart directly at the monkey. The monkey lets go at the instant the dart leaves the gun. Show that the dart will always hit the monkey, provided that the dart reaches the monkey before he hits the ground and runs away.

26 The tranquilizer dart hits the falling monkey.



We can easily show this using the SUVAT equations. The displacement of the monkey is:

$$y_{monkey} = d\tan\alpha_0 - \frac{1}{2}gt^2$$

And the displacement (y-component) of the dart is:

$$y_{dart} = (v_0 \sin\alpha_0)\, t - \frac{1}{2}gt^2$$

Considering the dart's x-component of displacement,

$$x_{dart} = (v_0 \cos\alpha_0)\, t$$

The time for the dart reach the other end is when $x_{dart} = d$, so

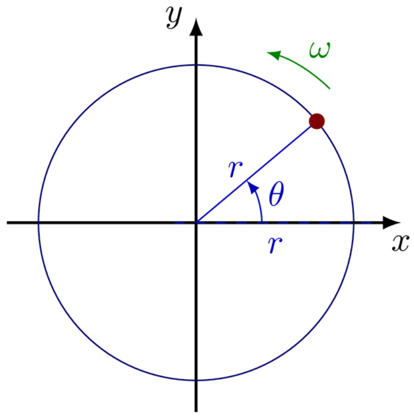
$$t_f = \frac{d}{v_0 \cos\alpha_0}$$

Substituting $t = t_f$ to the two equations above, we find that $y_{monkey} = y_{dart}$.

There's actually a way to see why the dart always hits the monkey, *without* any math! We'll revisit this example when we talk about frames of reference.

1.4 Circular Motion

As its name suggests, in circular motion, a particle A special case is uniform circular motion, where a particle travels in a **circle of constant radius and constant speed.** However, the particle **does not have constant velocity** as its direction is constantly changing!



Some basic relations you should know from H2 physics are

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{15}$$

$$v = r\omega \tag{16}$$

$$a_\theta = r\alpha \tag{17}$$

$$a_r = \frac{v^2}{r} = r\omega^2 \tag{18}$$

where the subscripts r and θ represent the radial and tangential acceleration. In H2 physics, you might call the former the centripetal acceleration.

You should also be comfortable dealing with the differential forms of quantities:

$$\omega = \frac{d\theta}{dt} \tag{19}$$

$$\alpha = \frac{d\omega}{dt} \tag{20}$$

When quantities are not constant with time, you'll need to fall back on Equations (19) and (20), and either simply differentiate, or separate variables and integrate.

Example 1.5 (Ricardo)**.** A particle moves in a circular path of radius R. At $t = 0$, it is located at $\theta_0 > 0$. The angle the particle makes with the vertical is given by

$$\theta(t) = \theta_0 - Bt^2,\ 0 < t < t_1,\ B > 0$$

Would the values of the tangential and radial components of the acceleration for $0 < t < t_1$ be zero, positive or negative?

This is quite a simple application of Equations (19) and (20). We first find ω:

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}\left(\theta_0 - Bt^2\right) = -2Bt$$

and then we find α:

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(-2Bt\right) = -2B$$

Knowing that $B < 0$ and $t > 0$, we conclude $a_r = R\omega^2 = R\left(-2Bt\right)^2 = 4RB^2t^2$ will always be positive, and $a_\theta = R\alpha = -2RB$ will always be negative.

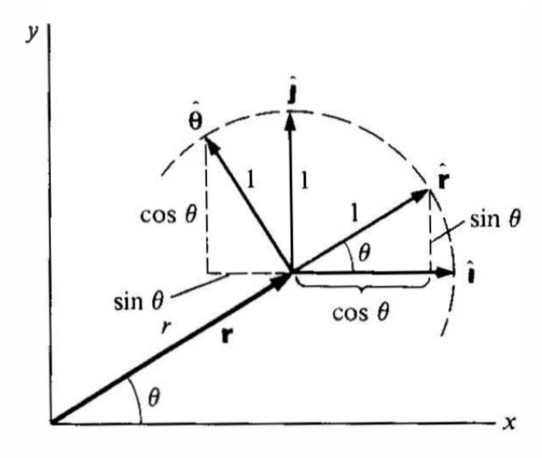
In non-uniform circular motion, we need to consider differential equations to track the acceleration in the radial and tangential components. More of this is covered in Section 1.5.1.

1.5 Ideas

Many tricky mechanics problems involve the use of the following ideas.

1.5.1 Polar Coordinates

Polar coordinates is a different coordinate system that is advantageous in many situations. Instead of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, we use $\hat{\mathbf{r}}$ and $\hat{\theta}$ for unit vectors in polar coordinates. It should be noted that these unit vectors are changing with respect to time, as an object moves.



The unit vectors always points in the increasing direction of magnitude. For example, $\hat{\mathbf{r}}$ points along $\mathbf{r}$ and $\hat{\theta}$ points perpendicular to $\hat{\mathbf{r}}$ in the increasing direction of θ. You can convert between coordinate systems with some trigonometry:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right) \tag{21}$$

$$x = r\cos\theta, \quad y = r\sin\theta \tag{22}$$

Or, with the vector formalism,

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}, \quad \hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}} \tag{23}$$

To represent velocity and acceleration in polar coordinates, we always consider the radial component and tangential components. However, this time, note that **r is changing with time!** Your equations will look different from Equations (16) to (20) because of this.

$$v_r = \dot{r}, \quad v_\theta = r\dot{\theta} \tag{24}$$

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{25}$$

When applying the acceleration formulas above, forces contributing to positive a_r point outward, and forces contributing to positive a_θ point to increase θ. Don't worry if this is confusing for now. The next example illustrates how to use this, and we'll go through how the forces work in later handouts.

Remark. We can prove this with the vector formalism (optional). Interested readers can refer to the Appendix to see how this is done.

Example 1.6. A particle moves at a constant $\dot{\theta} = \omega$ and with $r = r_0 e^{\alpha t}$. Show that for some value(s) of α, $a_r = 0$, and find the value(s).

This is a mainly mathematical example to get you used to manipulating the differential equations for acceleration in polar coordinates. Since we have r as a function of time,

$$\dot{r} = \alpha r_0 e^{\alpha t}, \ddot{r} = \alpha^2 r_0 e^{\alpha t}$$

Plugging these into Equation (25), noting that $\ddot{\theta} = 0$, we have,

$$a_r = \ddot{r} - r\dot{\theta}^2 = \alpha^2 r_0 e^{\alpha t} - \omega^2 r_0 e^{\alpha t}$$

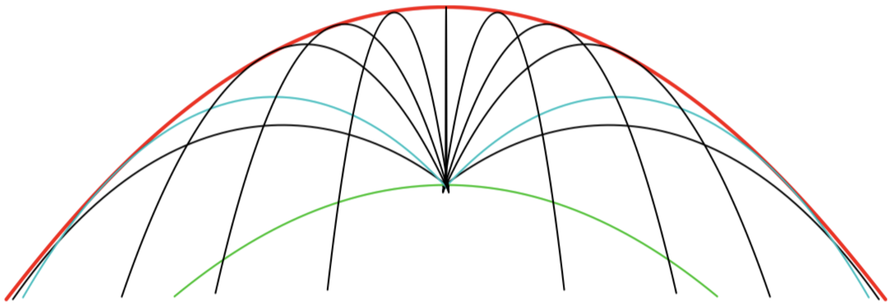
$$\implies a_r = 0 \quad \text{when} \quad \alpha = \pm\omega.$$

It may be surprising that a_r can be zero even though the radial velocity $\dot{r} = \alpha r_0 e^{\alpha t}$ is increasing with time — an apparent paradox. This is a consequence of the changing directions of the basis vectors. The apparent paradox arises from the misconception that a_r solely contributes to $\ddot{r}$, which is not true as we have neglected the centripetal acceleration!

1.5.2 Projectile Envelope

An envelope is defined to be the equation of the boundary of the region of space that a projectile can reach, given a fixed initial launch speed v_0 but a freely-varied launch angle θ from the horizontal. The equation of the envelope is quoted below.

$$y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} \tag{26}$$



The red curve is the envelope, the cyan curves are the trajectories for 45° launching angles, and the green curves are trajectories for 0° launching angles.

There is a fun, "almost no physics" way of deriving this. **All you need to memorise is that the equation of the envelope is a parabola, in 2D** (or a paraboloid, in 3D), and thus must take on the form $y = ax^2 + bx + c$. Now, we do some logical analysis.

1. The envelope is symmetric about $x = 0$ as the projectile starts at the origin. Thus, $b = 0$.
2. The envelope must be concave downwards as it doesn't make sense to extend the reachable area to positive infinity with a finite v_0. Thus, $a < 0$.
3. When the launching angle is $\theta = 90°$, the projectile goes straight up, and this must correspond to the maximum height of the envelope. The maximum height reached by such a projectile is given by $y_{max} = \frac{v_0^2}{2g}$ by simple kinematics. Thus, when $x = 0$, $y = \frac{v_0^2}{2g}$. This means that $c = \frac{v_0^2}{2g}$.
4. When the launching angle is $\theta = 45°$, you get the maximum range $x_{max} = \frac{v_0^2}{g}$ (if you don't already know this, prove it!). At this point, $y = 0$ when $x = \frac{v_0^2}{g}$. Thus, using the information so far about b and c, we have $0 = \frac{v_0^2}{2g} + a\left(\frac{v_0^2}{g}\right)^2$, giving $a = -\frac{g}{2v_0^2}$. Notice $a < 0$, which is consistent with our intuition in step 2.

Just like that, we have derived the equation of the projectile envelope, without any physics beyond simple kinematics. This hopefully also shows you the power of considering extreme cases and logical deduction.

The projectile envelope is a powerful way of finding **optimal projectile launch angles.**

Example 1.7. A projectile is shot from the origin with the **smallest possible launching speed** that allows it to hit a target on the ground some distance away. Show that the launching velocity is perpendicular to the final velocity (i.e. at the target).

To prove that they are perpendicular, we have to show that:

$$\left.\frac{dy}{dx}\right|_{x=0} \times \left.\frac{dy}{dx}\right|_{x=x'} = -1$$

since $\frac{dy}{dx}$ is the slope of the tangent to the trajectory, and the velocity is parallel to it. Here, x' is the position of the target.

Equation (26) defines the envelope, and the trajectory of the projectile is defined by

$$y = x\tan\theta - \frac{g}{2v_0^2\cos^2\theta}x^2 \tag{27}$$

Differentiating Equation (26),

$$\left.\frac{dy}{dx}\right|_{x=x'} = -\frac{gx'}{v_0^2}$$

Note that we are evaluating $\frac{dy}{dx}$ of the envelope at x', the position of the target. Since the projectile barely hits the target, we can think of this as an "extreme" case. The trajectory of the projectile must hence exactly cut the envelope when it hits the target. (The cyan curves in the image of the envelope shows this.)

Combining Equation (26) with (27) and performing some algebra,

$$\frac{v_0^2}{2g} - \frac{gx'^2}{2v_0^2} = x'\tan\theta - \frac{gx'^2}{2v_0^2\cos^2\theta} \quad\Rightarrow\quad \frac{gx'^2}{2v_0^2}\tan^2\theta - x'\tan\theta + \frac{v_0^2}{2g} = 0$$

$$\tan\theta = \frac{v_0^2}{gx'} = \left.\frac{dy}{dx}\right|_{x=0}$$

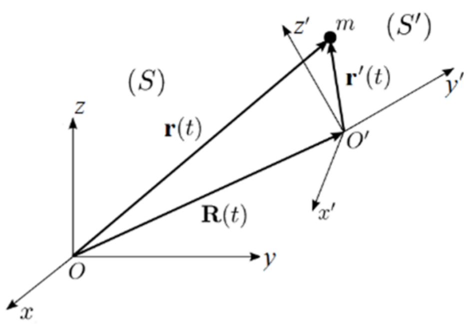
since $\tan\theta$ just gives the slope of the trajectory at the launching origin.

It is clear we have satisfied the condition to show that the initial velocity and final velocity are perpendicular. There exist a simpler geometric solution; can you find it yourself?

1.5.3 Frames of Reference

When many things are moving at once, it might be useful to go into a frame whereby the motion of some objects are **static or symmetric.**

Let frame S be an inertial (non-accelerating) frame (we usually call this the lab frame). Let frame S' have an origin O' located at the position vector $\mathbf{R}$ from the origin O of frame S.



From this, we have the following transformations between frames:

1. If a particle is at position $\mathbf{r}$ in frame S, it is at position $\mathbf{r}' = \mathbf{r} - \mathbf{R}$ in frame S', by simple vector addition.
2. Taking the derivative with respect to time, if a particle has velocity $\mathbf{u}$ in frame S, and frame S' moves at velocity $\mathbf{v}$ with respect to S, then it has velocity $\mathbf{u}' = \mathbf{u} - \mathbf{v}$ in frame S'.
3. Taking the derivative again, $\mathbf{a}' = \mathbf{a} - \mathbf{a}_{frame}$.

For most cases you will deal with in SPhO, S' will also be an inertial frame. This means that $\mathbf{a} = \mathbf{a}'$ (i.e. acceleration is invariant across inertial frames).

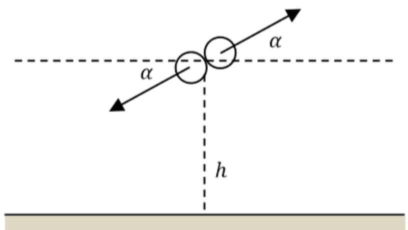
When we transform into a **non-inertial frame,** objects will now experience a **fictitious force**, since the acceleration in both frames are no longer equal. Newton's 2nd Law still holds, but one must now account for the fictitious acceleration.

1. For a **translationally** accelerating frame with acceleration $\mathbf{a}$, the fictitious force experienced by any object in this frame is $\mathbf{F}_{fic} = -m\mathbf{a}$, where m is the mass of the object. The negative sign here is important!
2. For a **rotating** frame, it involves the centrifugal, Coriolis, and Euler forces. This is far beyond the scope of SPhO, and you should never be using this.

Remark. This idea gives us some insight into Example 1.4. If we go into an accelerating frame with acceleration g downwards, the fictitious force term will be mg upwards where m could be the mass of the dart or monkey. Thus, there is no net force on both objects in this frame. The dart moves in a straight line directly towards the stationary monkey.

The following example illustrates the power of choosing the right frame of reference.

Example 1.8 (Ricardo)**.** Two balls start moving from the same height h from the ground. One is thrown with speed v_0 making an elevation angle α, the other is thrown with speed v_0 to the opposite direction from the first one, making a depression angle α. Calculate the distance between the 2 balls as a function of time, when both are still in the air.



While it is theoretically possible to consider the projectile motion of both balls and write their x and y displacements, this is unnecessary since you know about frames of reference.

The easiest one to work in is the frame accelerating at g downwards. In this frame, the balls simply move away in a straight line opposite to one another, thus the distance as a function of time is simply

$$D(t) = 2v_0 t$$

If you aren't convinced, try writing out the projectile motion equations for each axis to find x and y of each ball as a function of time. You'll end up doing at least 10 lines of math, while the solution above uses 1 line of math!

1.5.4 Optimisation

In many problems, you might be asked to find the maximum or minimum of some physical quantity. Mathematically, you can always express the quantity to be optimised as a function of a free variable, and then perform differentiation to determine the maximum or minimum. However, this is often tedious - there are other ways.

For one, we can use Fermat's principle. In essence, light takes the shortest path. To apply Fermat's principle to contexts besides light, we use Snell's law.

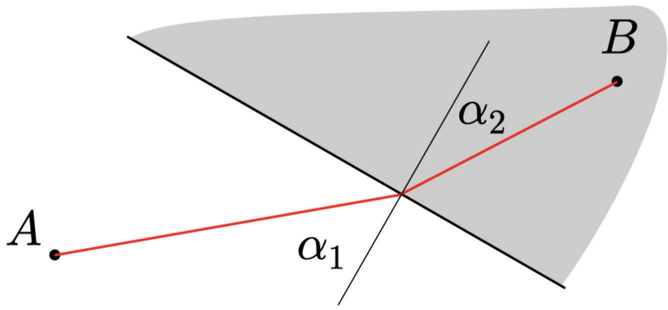


Figure 1: Snell's law

Let a point A be situated in a medium where the light propagation speed is v_1, and point B — in a medium where the speed is v_2. Then, the light propagates from A to B according to the Snell's law:

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{v_1}{v_2} \tag{28}$$

Lastly, you might encounter geometric ways to optimize physical quantities. If such a method is the intended solution, basic knowledge of properties of straight lines, triangles, circles and ellipses are sufficient.

1.5.5 Change of Variables Trick

In the Equation (11), the second derivative was strangely manipulated. To see how it arises, consider

$$\frac{d^2x}{dt^2} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx} \times \frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx} = \frac{1}{2}\frac{d(\dot{x})^2}{dx} \tag{29}$$

These two different ways to express second derivatives are **extremely useful** in Olympiad Physics. Make sure to remember this trick when solving differential equations, **especially if a time dependence is not required!**

1.6 References

An Introduction to Mechanics by *Kleppner & Kolenkow*

Kinematics Handout by *Jaan Kalda*

Competitive Physics by *Wang & Ricardo*

University Physics by *Young & Freedman*

An Introduction to Classical Mechanics by *David Morin*

2 Problems

Problems are arranged in roughly increasing difficulty.

Problem 2.1. At the instant a traffic light turns green, an automobile starts moving from rest with acceleration 2.2 m/s^2. At the same instant a truck, traveling at constant speed 9.5 m/s overtakes and passes the automobile. (a) How far beyond the starting point will the automobile overtake the truck? (b) How fast will the car be traveling at that instant?

Solution. Let $x_c(t)$ be the position of the automobile (car) and $x_t(t)$ be the position of the truck at time t. Let the starting point be $x = 0$ and the time the traffic light turns green be $t = 0$. At $t = 0$, $x_c(0) = 0$ and $x_t(0) = 0$.

The automobile (car) starts from rest, so its initial velocity $v_{c0} = 0\,\text{m/s}$. It moves with constant acceleration $a_c = 2.2\,\text{m/s}^2$. The position of the car is given by the kinematic equation $x_c(t) = x_{c0} + v_{c0}t + \frac{1}{2}a_c t^2$:

$$x_c(t) = 0 + 0 \cdot t + \frac{1}{2}(2.2)t^2 = 1.1t^2$$

The velocity of the car is given by $v_c(t) = v_{c0} + a_c t$:

$$v_c(t) = 0 + 2.2t = 2.2t$$

The truck travels at a constant speed $v_t = 9.5\,\text{m/s}$, so its acceleration $a_t = 0\,\text{m/s}^2$. The position of the truck is given by $x_t(t) = x_{t0} + v_t t + \frac{1}{2}a_t t^2$:

$$x_t(t) = 0 + (9.5)t + 0 = 9.5t$$

(a) To find how far beyond the starting point the automobile will overtake the truck, we set their positions equal, $x_c(t) = x_t(t)$.

$$1.1t^2 = 9.5t$$
$$1.1t^2 - 9.5t = 0$$
$$t(1.1t - 9.5) = 0$$

This equation has two solutions for t:

1. $t = 0\,\text{s}$: This corresponds to the initial moment when both are at the starting line and the truck is passing the car.
2. $1.1t - 9.5 = 0 \implies 1.1t = 9.5 \implies t = \frac{9.5}{1.1}\,\text{s}$.

The second solution is the time when the automobile overtakes the truck. $t = \frac{9.5}{1.1} \approx 8.6363...\,\text{s}$. The distance from the starting point at this time can be found by substituting this value of t into either position equation. Using the truck's position equation $x_t(t) = 9.5t$:

$$x = 9.5t = 9.5\left(\frac{9.5}{1.1}\right) = \frac{9.5^2}{1.1} = \frac{90.25}{1.1} \approx 82.0454...\,\text{m}$$

Alternatively, using the car's position equation $x_c(t) = 1.1t^2$:

$$x = 1.1\left(\frac{9.5}{1.1}\right)^2 = 1.1\frac{9.5^2}{1.1^2} = \frac{9.5^2}{1.1} \approx 82.0454...\,\text{m}$$

Rounding to two significant figures (based on the given data $2.2\,\text{m/s}^2$ and $9.5\,\text{m/s}$), the distance is $x \approx 82\,\text{m}$.

(b) To find how fast the car will be traveling at that instant, we use the car's velocity equation $v_c(t) = 2.2t$ at the time $t = \frac{9.5}{1.1}$ s.

$$v_c\left(\frac{9.5}{1.1}\right) = 2.2\left(\frac{9.5}{1.1}\right)$$

Since $2.2 = 2 \times 1.1$:

$$v_c = (2 \times 1.1)\left(\frac{9.5}{1.1}\right) = 2 \times 9.5 = 19\,\text{m/s}$$

At the instant the car overtakes the truck, its speed will be $19\,\text{m/s}$.

Problem 2.2 (SPhO 2019)**.** (i) A projectile is launched with initial velocity v_0 angled at θ to the horizontal. It intersects 2 points at height h above the ground. Find the distance D between the 2 points. (ii) Let the initial velocity be 200 m/s. Given that θ has been adjusted to give the maximum range and $h = 100$ m, find the value of D. *Note: The original problem statement missed out $h = 100$ m, making it underdefined.*

Solution. (i) Writing the SUVAT equations along the x and y-axes,

$$x = (v_0 \cos\theta)\, t,\; h = (v_0 \sin\theta)\, t - \frac{1}{2}gt^2$$

Thus, the times where the intersections happen are given by

$$\frac{1}{2}gt^2 - (v_0 \sin\theta)\, t + h = 0 \quad\Rightarrow\quad t = \frac{v_0 \sin\theta \pm \sqrt{v_0^2 \sin^2\theta - 2gh}}{g}$$

The difference in times is

$$\Delta t = t_2 - t_1 = \frac{2\sqrt{v_0^2 \sin^2\theta - 2gh}}{g}$$

As the horizontal velocity $v_x = v_0 \cos\theta$ is constant, the distance between the 2 points is

$$D = v_x \Delta t = \frac{2v_0 \cos\theta \sqrt{v_0^2 \sin^2\theta - 2gh}}{g}$$

(ii) Maximum range occurs when $\theta = 45°$. Plugging in the values into the result,

$$D = 3870\,\text{m}$$

Problem 2.3 (Young)**.** A ball is thrown straight up from the edge of the roof of a building. A second ball is dropped from the roof 1.00 s later. You may ignore air resistance. (a) If the height of the building is 20.0m, what must the initial speed of the first ball be if both are to hit the ground at the same time? On the same graph, sketch the position of each ball as a function of time, measured from when the first ball is thrown. Consider the same situation, but now let the initial speed v_0 of the first ball be given and treat the height h of the building as an unknown. (b)What must the height of the building be for both balls to reach the ground at the same time (i) if v_0 is 6.0 m/s and (ii) if v_0 is 9.5 m/s ? (c) If v_0 is greater than some value v_{max}, a value of h does not exist that allows both balls to hit the ground at the same time. Solve for v_{max}. The value v_{max} has a simple physical interpretation. What is it? (d) If v_0 is less than some value v_{min}, a value of h does not exist that allows both balls to hit the ground at the same time. Solve for v_{min}. The value v_{min} also has a simple physical interpretation. What is it?

Solution. Apply constant acceleration equations to both objects. **Set Up:** Let $+y$ be upward, so each ball has $a_y = -g$. For the purpose of doing all four parts with the least repetition of algebra, quantities will be denoted symbolically. That is, let

$$y_1 = h + v_0 t - \frac{1}{2}gt^2, \quad y_2 = h - \frac{1}{2}g(t - t_0)^2.$$

In this case, $t_0 = 1.00\,\text{s}$.

Execute: (a) Setting $y_1 = y_2 = 0$, expanding the binomial $(t - t_0)^2$ and eliminating the common term $\frac{1}{2}gt^2$ yields $v_0 t = gt_0 t - \frac{1}{2}gt_0^2$. Solving for t:

$$t = \frac{\frac{1}{2}gt_0^2}{gt_0 - v_0} = \frac{t_0}{2}\left(\frac{1}{1 - v_0/(gt_0)}\right).$$

Substitution of this into the expression for y_1 and setting $y_1 = 0$ and solving for h as a function of v_0 yields, after some algebra,

$$h = \frac{1}{2}gt_0^2\left(\frac{\frac{1}{2}gt_0 - v_0}{gt_0 - v_0}\right)^2.$$

Using the given value $t_0 = 1.00\,\text{s}$ and $g = 9.80\,\text{m/s}^2$,

$$h = 20.0\,\text{m} = (4.9\,\text{m})\left(\frac{4.9\,\text{m/s} - v_0}{9.8\,\text{m/s} - v_0}\right)^2.$$

(a) This has two solutions, one of which is unphysical (the first ball is still going up when the second is released; see part (c)). The physical solution involves taking the negative square root before solving for v_0, and yields $8.2\,\text{m/s}$.

(b) The above expression gives for (i) $0.411\,\text{m}$ and for (ii) $1.15\,\text{km}$.

(c) As v_0 approaches $9.8\,\text{m/s}$, the height h becomes infinite, corresponding to a relative velocity at the time the second ball is thrown that approaches zero. If $v_0 > 9.8\,\text{m/s}$, the first ball can never catch the second ball.

(d) As v_0 approaches $4.9\,\text{m/s}$, the height approaches zero. This corresponds to the first ball being closer and closer (on its way down) to the top of the roof when the second ball is released. If $v_0 < 4.9\,\text{m/s}$, the first ball will already have passed the roof on the way down before the second ball is released, and the second ball can never catch up.

Problem 2.4 (SPhO 2018)**.** A particle X is projected with speed V in a direction which makes an angle of 30° from the horizontal. When this particle reaches the highest point of its trajectory, another particle Y is dropped from the roof of a tall building. The two particles collide at the base of the building. The particle Y takes a time of 0.17 s to fall past a 5.0 m tall window somewhere along the building. The base of the window is 50.0 m above the ground. Ignoring air resistance, find (i) the height of the building (ii) the value of V (iii) the distance of the point of projection of X from the foot of the building.

Solution. It is best to draw a diagram yourself to visualise what is happening.

(i) Consider the time when Y is moving past the window. We have

$$h_{window} = v_{top,window} t_{window} + \frac{1}{2}gt_{window}^2 \quad \Rightarrow \quad v_{top,window} = \frac{h_{window} - \frac{1}{2}gt_{window}^2}{t_{window}} = 28.6\,\text{m/s}$$

Before this, the particle moves through a height $(H-55)$ m between the top of the building and top of the window. Noting that $v_{top,building} = 0\,\mathrm{m/s}$, thus, we have

$$v_{top,window}^2 = v_{top,building}^2 + 2g\,(H-55) \quad \Rightarrow \quad H = \frac{v_{top,window}^2 - v_{top,building}^2}{2g} + 55 = 96.7\,\mathrm{m}$$

(ii) The time taken for Y to fall from the top to bottom of the building is

$$t_Y = \sqrt{\frac{2H}{g}} = 4.44\,\mathrm{s}$$

This is exactly half the time of flight of X, since Y is released when X is at the highest point of its trajectory. Thus, $t_X = 2t_Y = 8.88\,\mathrm{s}$.

Consider the motion in the y-direction for X. Then, for the time of flight,

$$0 = (V\sin\theta)\,t_X - \frac{1}{2}gt_X^2 \quad \Rightarrow \quad t_X = \frac{2V\sin\theta}{g},\ \text{rejecting } t_X = 0 \text{ as a trivial solution}$$

Finally, since $\theta = 30°$, we have

$$V = \frac{gt_X}{2\sin\theta} = 87.1\,\mathrm{m/s}$$

(iii) This is just the range of projectile motion of X. Thus,

$$D = (V\cos\theta)\,t_X = 670\,\mathrm{m}$$

Problem 2.5 (SPhO 2019)**.** (i) A projectile is fired with initial speed 50 m/s from the edge of a cliff which is at a vertical height of 100 m from sea-level. It hits a target which is at a horizontal distance of 300 m from the bottom of the cliff on the surface of the sea. What is the angle of inclination of the initial velocity of the projectile above the horizontal? (ii) Suppose at the instant of projection, the target begins moving away from the cliff at a constant speed of 10 m/s. If the angle of projection remains the same as in (i), what must the speed of projection be so that the projectile hits the target?

Solution. Again, it is best to draw a diagram.

(i) Writing the SUVAT equations for the x and y-directions,

$$x = (v_0\cos\theta)\,t$$

$$y = (v_0\sin\theta)\,t - \frac{1}{2}gt^2$$

Notice that we don't need a time dependence, so we eliminate it using $t = \frac{x}{v_0\cos\theta}$, to get

$$y = (v_0\sin\theta)\left(\frac{x}{v_0\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2 = x\tan\theta - \frac{gx^2}{2v_0^2}\sec^2\theta = x\tan\theta - \frac{gx^2}{2v_0^2}\left(1+\tan^2\theta\right)$$

Since $y = -100\,\mathrm{m}$, $x = 300\,\mathrm{m}$, $v_0 = 50\,\mathrm{m/s}$, this is a quadratic in $\tan\theta$. It is a good practice for you to grind out the math (and this is as grindy as it gets in SPhO), and you should get

$$\tan\theta = \frac{v_0^2}{gx} \pm \sqrt{\frac{v_0^4}{g^2x^2} + \frac{2v_0^2}{gx^2}y - 1} \quad \Rightarrow \quad \theta = 54.2° \text{ or } \theta = 17.4°$$

(ii) Unfortunately, this part simply falls into more tedious algebraic grinding.

Let the speed of the target be v. Accounting for it moving away, we now have

$$x + vt = (v_0\cos\theta)\,t$$

$$y = (v_0 \sin\theta)\, t - \frac{1}{2} g t^2$$

Follow the same steps in (i), and eliminate the time dependence using $t = \frac{x}{v_0 \cos\theta - v}$, to get

$$\frac{1}{2} g \left(\frac{x}{v_0 \cos\theta - v} \right)^2 - \frac{x v_0 \sin\theta}{v_0 \cos\theta - v} + y = 0$$

At this point, you can just use your calculator's quadratic equation solver. Using the 2 values of θ in (i), you should get

$$v_0 = 60.7\,\mathrm{m/s} \text{ or } v_0 = 52.6\,\mathrm{m/s}$$

If (ii) was too much to grind out the quadratic equation symbolically, that's alright. However, try to get comfortable with the amount of algebraic grinding in (i) for SPhO.

Problem 2.6 (SPhO 2008)**.** A point moves along the x-axis with an acceleration of $a(t) = kt^2$, where t is the time the point has been in motion, and k is a constant. Find the distance traveled as a function of time, $x(t)$, if the initial speed is u.

Solution. Invoking the definition of acceleration, we can solve the corresponding differential equation:

$$\frac{dv}{dt} = kt^2$$

$$\int_u^v dv = \int_0^t kt^2\, dt$$

$$v(t) = u + \frac{1}{3} kt^3$$

Again, invoking the definition of velocity, we integrate again:

$$\frac{dx}{dt} = u + \frac{1}{3} kt^3$$

$$\int_0^x dx = \int_0^t \left(u + \frac{1}{3} kt^3 \right) dt$$

$$x(t) = ut + \frac{1}{12} kt^4$$

Problem 2.7 (Ricardo)**.** A cyclist has an acceleration in the x-direction given by $a_x = b_1 t^2$ where $b_1 > 0$. At $t = 0$, the cyclist is located at $x = 0$ and starts from rest. At the same time, a ball, located along the path of motion of the bicycle at $x_{ball,0} = d$, is thrown vertically upwards with velocity v_{y0}. At some later time $t = t_1$, the cyclist catches the ball on the way down, at the same height it was thrown from. Ignoring air resistance, find b_1 in terms of d, v_{y0} and g, but without using t_1.

Solution. First, let's consider the motion of the ball (vertical). The ball is released at $t = 0$ with initial velocity v_{y0} upwards and comes back to the same position at $t = t_1$. Thus,

$$0 = v_{y0} t_1 - \frac{1}{2} g t_1^2 \quad \Rightarrow \quad t_1 = \frac{2 v_{y0}}{g}, \text{ rejecting } t_1 = 0$$

Next, let's consider the motion of the cyclist (horizontal). Recall that $a_x = \frac{dv_x}{dt}$, thus

$$\frac{dv_x}{dt} = b_1 t^2 \quad \Rightarrow \quad dv_x = b_1 t^2\, dt \quad \Rightarrow \quad \int_0^{v_x(t)} dv_x = \int_0^t b_1 t^2\, dt \quad \Rightarrow \quad v_x(t) = \frac{1}{3} b_1 t^3$$

Now, recall that $v_x = \frac{dx}{dt}$, so we integrate once more to get

$$\frac{dx}{dt} = \frac{1}{3} b_1 t^3 \quad \Rightarrow \quad dx = \frac{1}{3} b_1 t^3\, dt \quad \Rightarrow \quad \int_0^{x(t)} dx = \int_0^t \frac{1}{3} b_1 t^3\, dt \quad \Rightarrow \quad x(t) = \frac{1}{12} b_1 t^4$$

For the cyclist to catch the ball, $x(t) = d$, thus

$$\frac{1}{12}b_1 t_1^4 = d \quad \Rightarrow \quad b_1 = \frac{12d}{t_1^4} = \frac{12d}{\left(\frac{2v_{y0}}{g}\right)^4} = \frac{3g^4 d}{4v_{y0}^4}$$

Problem 2.8 (Ricardo)**.** At $t = 0$, you start to spin a bicycle wheel with your finger by applying a torque on one of the spokes. You stop applying the torque at $t = t_1$. The z-component of the angular acceleration is given by

$$\alpha_z(t) = \begin{cases} b\left(1 - \frac{t}{t_1}\right), \, 0 \le t \le t_1 \\ 0, \; t > t_1 \end{cases}$$

where $b > 0$. (i) Find the expression for the z-component of angular velocity $\omega_z(t)$ for $0 \le t \le t_1$. (ii) Find the expression for the angular displacement $\theta(t)$ for $0 \le t \le t_1$.

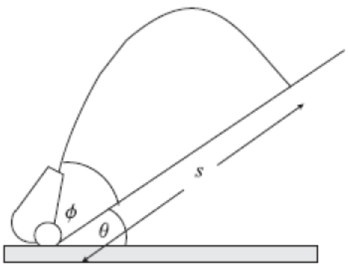
Solution. (i) Recalling that $\alpha = \frac{d\omega}{dt}$, thus

$$d\omega_z = \alpha_z \, dt \quad \Rightarrow \quad \int_0^{\omega_z(t)} d\omega_z = \int_0^t b\left(1 - \frac{t}{t_1}\right) dt \quad \Rightarrow \quad \omega_z(t) = b\left[t - \frac{t^2}{2t_1}\right]_0^t = b\left(t - \frac{t^2}{2t_1}\right)$$

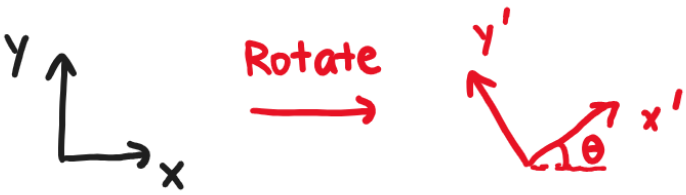
(ii) Recalling that $\omega = \frac{d\theta}{dt}$, thus

$$d\theta = \omega_z \, dt \quad \Rightarrow \quad \int_0^{\theta(t)} d\theta = \int_0^t b\left(t - \frac{t^2}{2t_1}\right) dt \quad \Rightarrow \quad \theta(t) = b\left[\frac{t^2}{2} - \frac{t^3}{6t_1}\right]_0^t = b\left(\frac{t^2}{2} - \frac{t^3}{6t_1}\right)$$

Problem 2.9 (SPhO 2004)**.** A cannon fires a shot at an angle of φ up a plane inclined to the horizontal at angle θ as shown. In terms of θ, at what angle φ should the shot be fired in order to maximise the distance s up the inclined plane?



Solution. The easiest way to avoid long, tedious equations is to rotate coordinate axes.



Along the x'-axis, the effective acceleration is $a_{x'} = -g\sin\theta$, and along the y'-axis, the effective acceleration is $a_{y'} = -g\cos\theta$. You can find this by projecting $\mathbf{g}$ (which points vertically downwards) onto the x' and y'-axes.

Thus, the two equations of motion are

$$x' = (v_0 \cos\varphi)\, t - \frac{1}{2}(g \sin\theta)\, t^2$$

$$y' = (v_0 \sin\varphi)\, t - \frac{1}{2}(g \cos\theta)\, t^2$$

When the projectile lands on the plane, $y' = 0$. Rejecting $t = 0$ as a trivial solution, we have

$$v_0 \sin\varphi = \frac{1}{2}(g\cos\theta)\, t \quad \Rightarrow \quad t = \frac{2v_0 \sin\varphi}{g\cos\theta}$$

Substituting this back into x' (which is the distance up the inclined plane),

$$x' = (v_0\cos\varphi)\frac{2v_0\sin\varphi}{g\cos\theta} - \frac{1}{2}g\sin\theta\left(\frac{2v_0\sin\varphi}{g\cos\theta}\right)^2 = \frac{2v_0^2}{g\cos\theta}\left(\sin\varphi\cos\varphi - \frac{\sin\theta\sin^2\varphi}{\cos\theta}\right)$$

To maximise x', we maximise the term inside the brackets, since the term outside is constant. Thus, we take the derivative with respect to φ and set it equal to 0,

$$\frac{d}{d\varphi}\left(\sin\varphi\cos\varphi - \frac{\sin\theta\sin^2\varphi}{\cos\theta}\right) = \frac{d}{d\varphi}\left(\frac{1}{2}\sin(2\varphi) - \tan\theta\sin^2\varphi\right) = \cos(2\varphi) - \sin(2\varphi)\tan\theta = 0$$

$$\tan(2\varphi) = \frac{1}{\tan\theta} = \tan\left(\frac{\pi}{2} - \theta\right) \quad \Rightarrow \quad 2\varphi = \frac{\pi}{2} - \theta \quad \Rightarrow \quad \varphi = \frac{\pi}{4} - \frac{\theta}{2}$$

Problem 2.10 (SPhO 2018)**.** A particle of mass 100 g is dropped from a great height and falls vertically downwards. The force due to air resistance is $-kv$, where v is the speed of the particle and $k = 1.09 \times 10^{-2}$ kg/s. (i) What is the terminal velocity of the particle? (ii) How much time does it take for the particle to achieve 99% of the terminal velocity? (iii) How far has the particle fallen at this time?

Solution. (i) Terminal velocity occurs when the air resistance perfectly cancels the gravitational force. Thus,

$$kv_T = mg \quad \Rightarrow \quad v_T = \frac{mg}{k} = 90\,\text{m/s}$$

(ii) The general equation of motion before terminal velocity is

$$m\frac{dv}{dt} = mg - kv \quad \Rightarrow \quad \frac{m\,dv}{mg - kv} = dt \quad \Rightarrow \quad \int_0^{0.99v_T} \frac{m\,dv}{mg - kv} = \int_0^{t_{req}} dt$$

Thus, the required time is

$$t_{req} = -\frac{m}{k}\left[\ln(mg - kv)\right]_0^{0.99v_T} = -\frac{m}{k}\left(\ln\left(\frac{mg - 0.99kv_T}{mg}\right)\right) = 42.3\,\text{s}$$

(iii) Since we don't need a time dependence now, from the equation of motion,

$$m\frac{dv}{dt} = mv\frac{dv}{dx} = mg - kv \quad \Rightarrow \quad \frac{mv\,dv}{mg - kv} = dx \quad \Rightarrow \quad \int_0^{0.99v_T}\frac{mv\,dv}{mg - kv} = \int_0^{x_{req}} dx$$

Thus, the required distance is

$$x_{req} = \int_0^{0.99v_T}\frac{mv\,dv}{mg - kv} = m\int_0^{0.99v_T}\left(\frac{v - \frac{mg}{k} + \frac{mg}{k}}{mg - kv}\right)dv = m\int_0^{0.99v_T}\left(-\frac{1}{k} + \frac{mg}{k}\frac{1}{mg - kv}\right)dv$$

$$= m\left[-\frac{v}{k} - \frac{mg}{k^2}\ln(mg - kv)\right]_0^{0.99v_T} = -m\left(\frac{0.99v_T}{k} + \frac{mg}{k^2}\ln(mg - 0.99kv_T)\right) = 3000\,\text{m}$$

Problem 2.11 (Ricardo)**.** In a straight river of width $2a$, the water flows parallel to the banks with speed $v(y) = V_0\left(1 - \frac{y^2}{a^2}\right)$, where y is measured from the center of the river. (This is a parabolic velocity profile, typical of fluid flow following the Hagen-Poiseuille equation, which will be covered in later handouts.) A man, who can maintain a speed V in still water, swims across the river directly towards the opposite bank. How much has he moved in the x-direction when he reaches the opposite bank?

Solution. Set up the axes so that the x-axis is parallel to the river banks and the y-axis is perpendicular to the river banks. First, you should realise that the velocity profile is symmetric about $y = 0$. Thus, the drift in the x-direction caused by $y = -a$ to $y = 0$ is the same as that caused by $y = 0$ to $y = a$.

Consider $y = 0$ to $y = a$. In still water,

$$v_y = V = \frac{dy}{dt} \quad \Rightarrow \quad dt = \frac{dy}{V}$$

Along the x-axis,

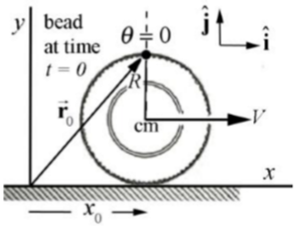
$$dx = v\,dt = v\,\frac{dy}{V} = V_0\left(1 - \frac{y^2}{a^2}\right)\frac{dy}{V}$$

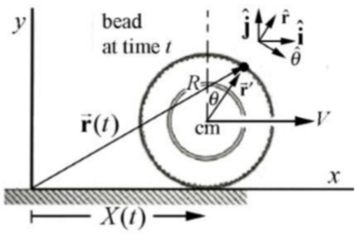
Thus,

$$\int_0^{\frac{x_{drift}}{2}} dx = \int_0^a \frac{V_0}{V}\left(1 - \frac{y^2}{a^2}\right) dy \quad \Rightarrow \quad \frac{x_{drift}}{2} = \frac{V_0}{V}\left[y - \frac{y^3}{3a^2}\right]_0^a$$

$$x_{drift} = \frac{2V_0}{V}\left(a - \frac{a}{3}\right) = \frac{2V_0}{V}\left(\frac{2a}{3}\right) = \frac{4aV_0}{3V}$$

Problem 2.12 (Ricardo)**.** A bicycle wheel of radius R is rolling without slipping along a horizontal surface. The centre of mass of the wheel moves at a constant speed V in the positive x-direction. A bead is lodged onto the rim of the wheel. At $t = 0$, the bead is located at the top of the wheel at $(x_0, 2R)$. What are the x and y-components of the position of the bead as a function of time according to an observer fixed to the ground?





Solution. Don't be intimidated by the rolling, this is actually just an exercise about frames of reference! Go into the frame moving with the wheel. The wheel is stationary in this frame.

In this frame, the bead is located at

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} R\sin\theta \\ R\cos\theta \end{pmatrix}$$

where θ is as defined in the figure above.

The rotation of the wheel means that

$$\omega = \frac{d\theta}{dt} \quad \Rightarrow \quad d\theta = \omega\,dt \quad \Rightarrow \quad \int_0^{\theta(t)} d\theta = \int_0^t \omega\,dt \quad \Rightarrow \quad \theta(t) = \omega t$$

Recall from your H2 physics that for rolling without slipping,

$$V = R\omega \quad \Rightarrow \quad \theta(t) = \frac{Vt}{R}$$

The center of mass of the wheel has position as a function of time

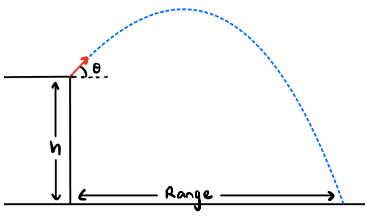
$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \begin{pmatrix} x_0 + Vt \\ R \end{pmatrix}$$

Thus, in the lab frame (observer fixed to the ground), the position of the bead is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_c \\ y_c \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x_0 + Vt + R\sin\theta \\ R + R\cos\theta \end{pmatrix} = \begin{pmatrix} x_0 + Vt + R\sin\left(\frac{Vt}{R}\right) \\ R + R\cos\left(\frac{Vt}{R}\right) \end{pmatrix}$$

You may verify that when $\theta = 0$ and $t = 0$, the position of the bead is $(x_0, 2R)$, as desired.

Problem 2.13. A projectile is launched off a cliff of height h relative to the ground level at an angle θ. For what angle θ_{max} will the horizontal range be maximized? Hint: there is a neat geometrical proof that avoids tedious calculations.



Solution. The answer is $\theta_{max} = \arctan\left(\frac{v_o}{\sqrt{v_o + 2gh}}\right)$. The neat solution is illustrated here.

Problem 2.14. A projectile of mass m is dropped from an initial height h above the ground. There is quadratic drag present, $F = -\alpha m v^2$. It falls and bounces elastically back up. Find the maximum height to which it subsequently rises. Hint: integrating the expression $\frac{dv}{dt}$ may not be the most efficient way to do it, since the question is not asking for $v(t)$ explicitly. In fact, the question doesn't need a time dependence at all!

Solution. During the downward motion, Newton's 2nd Law gives

$$m\frac{dv}{dt} = mg - \alpha m v^2 \quad \Rightarrow \quad \frac{dv}{dt} = g - \alpha v^2$$

As mentioned, a time dependence is not required, so we remove it using the chain rule, by writing $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$. Thus,

$$v\frac{dv}{dx} = g - \alpha v^2 \quad \Rightarrow \quad \int_0^h dx = \int_0^{v_f} \frac{v}{g - \alpha v^2}\, dv \quad \Rightarrow \quad h = \int_0^{v_f} \frac{v}{g - \alpha v^2}\, dv$$

Let $u = g - \alpha v^2$. Then, $\frac{du}{dv} = -2\alpha v$. Thus,

$$h = \int_g^{g - \alpha v_f^2} \frac{v}{u} \frac{du}{-2\alpha v} = \int_{g - \alpha v_f^2}^{g} \frac{1}{2\alpha u}\, du = \frac{1}{2\alpha} \ln\left(\frac{g}{g - \alpha v_f^2}\right) \quad \Rightarrow \quad v_f^2 = \frac{g}{\alpha}\left(1 - e^{-2\alpha h}\right)$$

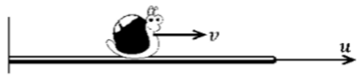
During the upward motion, Newton's 2nd Law gives

$$m\frac{dv}{dt} = -mg - \alpha m v^2 \quad \Rightarrow \quad \frac{dv}{dt} = -g - \alpha v^2$$

Using a similar procedure and integrating, and now noting that v_f is the starting upward velocity (since the collision is elastic), we get

$$h_{max} = \frac{1}{2\alpha} \ln\left(\frac{g + \alpha v_f^2}{g}\right) = \frac{1}{2\alpha} \ln\left(1 + \frac{\alpha}{g}\left(\frac{g}{\alpha}\right)\left(1 - e^{-2\alpha h}\right)\right) = \frac{1}{2\alpha} \ln\left(2 - e^{-2\alpha h}\right)$$

Problem 2.15 (Ricardo)**.** The left end of an elastic chord of initial length L_0 is attached to a fixed wall and its right end is pulled with a constant velocity u to the right by a spider. At $t = 0$, without knowing the existence of the spider, a snail starts moving from the wall with constant velocity v *relative to the chord*. When will the snail reach the other end of the chord?



Solution. The elastic chord lengthens with time. The length of the chord at time t is

$$L(t) = L_0 + ut$$

Clearly, the speed at points along the chord should vary linearly from 0 at the left end to u at the right end. Thus, at distance x (where $0 \leq x \leq L$) from the left end,

$$v_{chord} = \frac{x}{L}u = \frac{ux}{L_0 + ut}$$

The snail's speed in the lab frame is thus

$$v_{snail} = v + v_{chord} = v + \frac{ux}{L_0 + ut} = \frac{dx}{dt} \quad \Rightarrow \quad \frac{dx}{dt} - \frac{u}{L_0 + ut}x = v$$

To solve this first-order ODE, recall that we can use an integrating factor

$$I(x) = e^{\int -\frac{u}{L_0+ut}\,dt} = e^{-u\left(\frac{1}{u}\right)\ln(L_0+ut)} = \frac{1}{L_0 + ut}$$

Multiplying $I(x)$ to both sides, we have

$$\frac{d}{dt}\left(\frac{x}{L_0 + ut}\right) = \frac{v}{L_0 + ut} \quad \Rightarrow \quad \frac{x}{L_0 + ut} = \int \frac{v}{L_0 + ut}\,dt = \frac{v}{u}\ln\left(L_0 + ut\right) + C$$

As the snail starts moving from the left end, $x = 0$ when $t = 0$, thus

$$0 = \frac{v}{u}\ln L_0 + C \quad \Rightarrow \quad C = -\frac{v}{u}\ln L_0$$

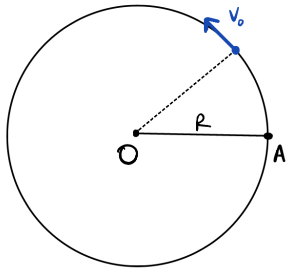
Thus,

$$\frac{x}{L_0 + ut} = \frac{v}{u}\ln\left(\frac{L_0 + ut}{L_0}\right) \quad \Rightarrow \quad x = \frac{(L_0 + ut)\,v}{u}\ln\left(\frac{L_0 + ut}{L_0}\right)$$

The snail reaches the other end when $x = L = L_0 + ut$, thus

$$1 = \frac{v}{u}\ln\left(\frac{L_0 + ut}{L_0}\right) \quad \Rightarrow \quad t = \frac{L_0}{u}\left(e^{\frac{u}{v}} - 1\right)$$

Problem 2.16. A rat runs with uniform speed v_0 along a circle of radius R. When it passes point A, a cat sets off at point O and chases after it with the same speed. Given that during the chase, point O, the cat and the rat remain on one straight line, find the path taken by the cat and the time at which the cat catches the rat.



Solution. Let the origin O be the center of the circle. Let the rat start at point A on the circle, which we can place at $(R, 0)$ in Cartesian coordinates, or $r = R, \theta = 0$ in polar coordinates at $t = 0$. The rat runs with uniform speed v_0 along the circle of radius R. The angular speed of the rat is $\omega = \frac{v_0}{R}$. At time t, the angular position of the rat is $\theta_r(t) = \omega t = \frac{v_0}{R}t$. The position of the rat in polar coordinates is $(R, \theta_r(t))$.

The cat starts from point O at $t = 0$ and chases the rat with the same speed v_0. It is given that during the chase, point O, the cat, and the rat remain on one straight line. This means that at any time t, the angular position of the cat, $\theta_c(t)$, is the same as the angular position of the rat, $\theta_r(t)$. Let this common angle be $\theta(t) = \frac{v_0}{R}t$. Let the cat's radial distance from O at time t be $r_c(t)$. The position of the cat in polar coordinates is $(r_c(t), \theta(t))$.

The velocity of the cat in polar coordinates has two components:

1. Radial velocity: $v_r = \frac{dr_c}{dt}$
2. Tangential velocity: $v_\theta = r_c \frac{d\theta}{dt}$

We know $\frac{d\theta}{dt} = \omega = \frac{v_0}{R}$. So, $v_\theta = r_c \left(\frac{v_0}{R}\right)$. The magnitude of the cat's speed is v_0. Therefore,

$$v_0^2 = \left(\frac{dr_c}{dt}\right)^2 + \left(r_c \frac{d\theta}{dt}\right)^2$$

$$v_0^2 = \left(\frac{dr_c}{dt}\right)^2 + \left(r_c \frac{v_0}{R}\right)^2$$

$$v_0^2 = \left(\frac{dr_c}{dt}\right)^2 + r_c^2 \frac{v_0^2}{R^2}$$

Divide by v_0^2:

$$1 = \frac{1}{v_0^2}\left(\frac{dr_c}{dt}\right)^2 + \frac{r_c^2}{R^2}$$

$$\left(\frac{dr_c}{dt}\right)^2 = v_0^2 \left(1 - \frac{r_c^2}{R^2}\right)$$

Since the cat is moving away from the origin towards the rat, $\frac{dr_c}{dt} > 0$.

$$\frac{dr_c}{dt} = v_0 \sqrt{1 - \frac{r_c^2}{R^2}} = \frac{v_0}{R}\sqrt{R^2 - r_c^2}$$

This is a separable differential equation:

$$\frac{dr_c}{\sqrt{R^2 - r_c^2}} = \frac{v_0}{R} dt$$

Integrate both sides:

$$\int \frac{dr_c}{\sqrt{R^2 - r_c^2}} = \int \frac{v_0}{R} dt$$

The integral on the left is $\arcsin\left(\frac{r_c}{R}\right)$.

$$\arcsin\left(\frac{r_c}{R}\right) = \frac{v_0}{R}t + C$$

At $t = 0$, the cat is at O, so $r_c(0) = 0$.

$$\arcsin\left(\frac{0}{R}\right) = \frac{v_0}{R}(0) + C \implies \arcsin(0) = C \implies C = 0$$

So, the radial position of the cat as a function of time is:

$$\arcsin\left(\frac{r_c}{R}\right) = \frac{v_0}{R}t$$

$$r_c(t) = R\sin\left(\frac{v_0}{R}t\right)$$

The path taken by the cat is described by its radial position r_c as a function of its angular position θ. Since $\theta(t) = \frac{v_0}{R}t$, we can substitute this into the expression for $r_c(t)$:

$$r_c = R\sin(\theta)$$

This is the polar equation of a circle of diameter R, passing through the origin, with its diameter along the line $\theta = \pi/2$.

The cat catches the rat when the cat's radial position is equal to the radius of the rat's circular path, i.e., $r_c(t) = R$.

$$R = R\sin\left(\frac{v_0}{R}t\right)$$

$$1 = \sin\left(\frac{v_0}{R}t\right)$$

For the first time this occurs (smallest positive t):

$$\frac{v_0}{R}t = \frac{\pi}{2}$$

$$t = \frac{\pi R}{2v_0}$$

At this time, the angular position is $\theta = \frac{v_0}{R}t = \frac{v_0}{R}\frac{\pi R}{2v_0} = \frac{\pi}{2}$. The rat is at $(R, \pi/2)$ and the cat is at $r_c = R\sin(\pi/2) = R$, so the cat is also at $(R, \pi/2)$.

The path taken by the cat is given by the polar equation $r_c = R\sin(\theta)$. The time at which the cat catches the rat is $t = \frac{\pi R}{2v_0}$.

3 Advanced Problems

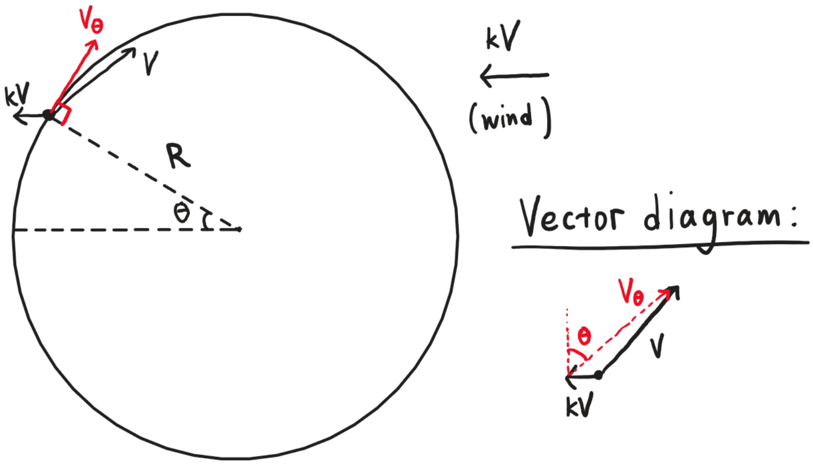
These problems are way too difficult to be tested in a modern-day SPhO. If you have completed all the previous problems and are down for a challenge, try these!

Problem 3.1 (SPhO 2006)**.** An aeroplane flies at a constant speed V *relative* to the air and completes a level circular course in time T on a windless day. If there is a steady wind of speed kV blowing in a fixed horizontal direction, determine the increase (or decrease) in the time needed for the course, assuming that $k << 1$, in terms of k and T. The aeroplane's speed changes along the course to ensure that it remains circular. (Hint: use $k << 1$ to make an approximation for an otherwise difficult integral.)

Solution. Let the circular course have radius R. Then, under windless conditions,

$$T = \frac{2\pi R}{V} \quad \Rightarrow \quad R = \frac{TV}{2\pi}$$

After the wind is blowing, draw a vector diagram:



By the cosine rule,

$$V^2 = (kV)^2 + V_\theta^2 - 2kVV_\theta \cos(90^\circ - \theta) \quad \Rightarrow \quad V_\theta^2 - 2kVV_\theta \sin\theta + \left(k^2 - 1\right) V^2 = 0$$

Solving the quadratic equation for V_θ gives

$$V_\theta = kV \sin\theta \pm V\sqrt{1 - k^2 \cos^2\theta}$$

To decide which root to reject, notice that when $\theta = 0^\circ$, the negative root would give you $-V\sqrt{1-k^2}$. But, all our quantities are speeds - there is no way that they will be negative! Thus, we need to reject the negative root.

So, the integral for the time taken is

$$T_{wind} = \int dt = \int_0^{2\pi} \frac{R}{V_\theta}\, d\theta = \int_0^{2\pi} \frac{R}{V\left(k\sin\theta + \sqrt{1 - k^2\cos^2\theta}\right)}\, d\theta$$

After trying for a while, you may realise this integral is impossible to evaluate. In fact, they are called elliptic integrals - their solutions are non-elementary!

Thankfully, the good news is that $k << 1$. Thus,

$$\sqrt{1-k^2\cos^2\theta} \approx 1 - \frac{1}{2}k^2\cos^2\theta$$

Now, let $u = k\sin\theta - \frac{1}{2}k^2\cos^2\theta$. Using the Maclaurin series, $(1+u)^{-1} \approx 1 - u + u^2$, thus

$$\frac{1}{k\sin\theta + \sqrt{1-k^2\cos^2\theta}} \approx \frac{1}{1 + k\sin\theta - \frac{1}{2}k^2\cos^2\theta}$$

$$\approx 1 - \left(k\sin\theta - \frac{1}{2}k^2\cos^2\theta\right) + \left(k\sin\theta - \frac{1}{2}k^2\cos^2\theta\right)^2 \approx 1 - k\sin\theta + \frac{1}{2}k^2 + \frac{1}{2}k^2\sin^2\theta$$

ignoring all terms of order k^3 and above.

Thus, going back to the integral,

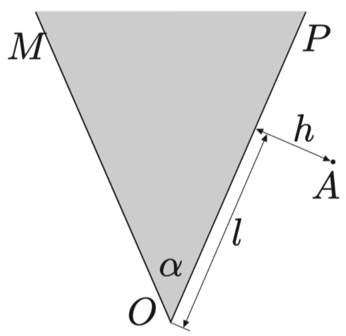
$$T_{wind} = \frac{R}{V}\int_0^{2\pi}\left(1 - k\sin\theta + \frac{1}{2}k^2 + \frac{1}{2}k^2\sin^2\theta\right)d\theta$$

$$= \frac{R}{V}\left[\theta + k\cos\theta + \frac{1}{2}k^2\theta + \frac{1}{2}k^2\left(\frac{1}{2}\theta - \frac{\sin(2\theta)}{4}\right)\right]_0^{2\pi} = \frac{2\pi R}{V} + \left(\frac{3}{4}k^2\right)\frac{2\pi R}{V} = T\left(1 + \frac{3}{4}k^2\right)$$

Thus, the **increase** in time taken is

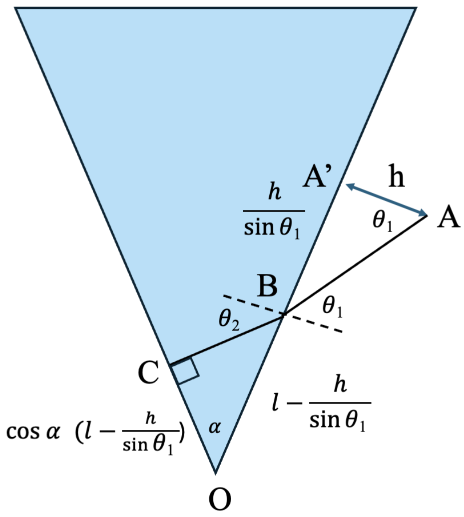
$$\Delta T = \frac{3}{4}k^2 T$$

Problem 3.2 (Kalda)**.** A boy lives on the shore OP of a bay MOP (see the figure). Two shores of the bay make an angle α. The boy's house is situated at point A at distance h from the shore and $\sqrt{h^2+l^2}$ from point O. The boy wants to go fishing to the shore OM. At what distance x from point O should be the fishing spot, so that it would take as little time as possible to get there from the house? How long is this time? The boy moves at velocity v on the ground and at velocity $u < v$ when using a boat.



Solution. The tricky part about this problem is that you have to find the geometric constraint and combine it with Snell's law. Only doing either one of them will lead to either intractable equations or the wrong answer!

Let's draw a diagram of a possible path.



Looking at the diagram above, we see that the geometric constraint so that we take as little time as possible is that $\angle BCO$ is $90°$. You should convince yourself why.

Some lengths are also labelled on the diagram. We see that CO, the length we want to minimise, is given by

$$CO = \cos\alpha \left(l - \frac{h}{\sin\theta_1} \right)$$

Now, we need to use Snell's law to eliminate the unknown variable θ_1,

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_1}{n_2} = \frac{c/v_1}{c/v_2} = \frac{v_2}{v_1}$$

Furthermore, from the geometric constraint, we have that $\alpha = \theta_2$. Hence:

$$\frac{\sin\theta_1}{\sin\alpha} = \frac{u}{v}$$

Using this, we can obtain the final solution,

$$CO = \cos\alpha \left(l - \frac{vh}{u\sin\alpha} \right)$$

Problem 3.3 (Kalda)**.** Two fences of heights h_1 and h_2 are erected on a horizontal plain, so that the *tops* of the fences are separated by d. Show that the minimum speed to throw a projectile over both fences is

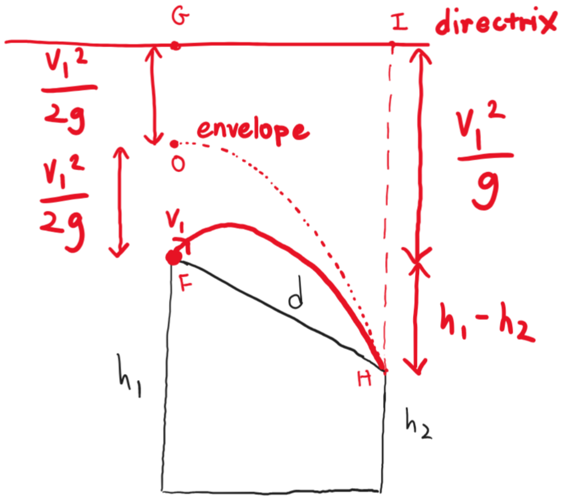
$$v_{min} = \sqrt{g\left(h_1 + h_2 + d\right)}$$

given that the position and angle of launch can *both* be freely varied. (Hint: The idea of the projectile envelope in Section 1.5.2, along with some geometric properties of parabolas involving focus and directrix, will be helpful.)

Solution. Let's split the problem up into some subproblems so that we can eventually apply the properties of the projectile envelope.

At minimum possible speed, the projectile should barely graze the top of both fences.

Consider the ball being "launched" from the top point of the first fence, in the figure below.



There are some important points when constructing this.

1. The actual trajectory of the projectile touches the projectile envelope at point H (the top of the second fence) - this is the minimality condition for the projectile to barely graze point H. Else, the projectile will be above point H as it passes through.
2. The point of launch is the *focus* of the parabola of the projectile envelope.
3. $OF = OG$, as G lies along the *directrix* of the parabola.
4. $FH = IH$, again due to directrix properties.

At this point, if you aren't familiar with parabola properties, you should find out before proceeding further.

As the maximum height of the projectile envelope is $\frac{v_1^2}{2g}$, thus $FG = \frac{v_1^2}{g}$. Thus,

$$IH = \frac{v_1^2}{g} + (h_1 - h_2) = FH = d \quad \Rightarrow \quad v_1^2 = g\left(d + h_2 - h_1\right)$$

Lastly, to find the launch speed from the ground, using conservation of energy,

$$\frac{1}{2}mv_{min}^2 = \frac{1}{2}mv_1^2 + mgh_1 \quad \Rightarrow \quad v_{min} = \sqrt{g\left(h_1 + h_2 + d\right)}$$

4 Appendix

4.1 Vector Formalism of Velocity and Acceleration in Polar Coordinates

Recall that

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}, \quad \hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}.$$

Differentiating with respect to time,

$$\dot{\hat{\mathbf{r}}} = -\sin\theta\,\dot{\theta}\hat{\mathbf{x}} + \cos\theta\,\dot{\theta}\hat{\mathbf{y}}, \quad \dot{\hat{\boldsymbol{\theta}}} = -\cos\theta\,\dot{\theta}\hat{\mathbf{x}} - \sin\theta\,\dot{\theta}\hat{\mathbf{y}}.$$

Comparing both equations, we can say that:

$$\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}} \quad \text{and} \quad \dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\mathbf{r}}.$$

For a general position vector, we can apply the product rule and invoke the previous result:

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

To find acceleration, we can take the 2nd derivative:

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}(-\dot{\theta}\hat{\mathbf{r}}) \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.\end{aligned}$$

Finally obtaining the acceleration:

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

These terms have physical meaning: the 2nd term of the radial acceleration represents the centripetal acceleration, while the 2nd term of the tangential acceleration represents the Coriolis acceleration. (The Coriolis acceleration is outside of even the IPhO syllabus! It arises purely from the mathematics.)